Homework 1

1. **Estimating logarithm function(15 points).** For $x \in [0, 1)$, we shall use the identity that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

   (a) **(5 points)** Prove that $\ln(1-x) \leqslant -x - \frac{x^2}{2}$.
   Solution.

(b) **(10 points)** For $x \in [0, 1/2]$, prove that

$$\ln(1-x) \geqslant -x - \frac{x^2}{2 \cdot 2^0} - \frac{x^2}{2 \cdot 2^1} - \frac{x^2}{2 \cdot 2^2} - \frac{x^2}{2 \cdot 2^3} - \cdots = -x - x^2.$$

Solution.

2. **Tight Estimations(25 points)** Provide meaningful upper-bounds and lower-bounds for the following expressions.

   (a) **(10 points)** $S_n = \sum_{i=1}^{n} \ln i$,
   Solution.

(b) **(5 points)** $A_n = n!$
Solution.

(c) **(10 points)** $B_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$
Solution.

3. **Understanding Joint Distribution(15 points)** Recall that in the lectures we considered the joint distribution $(\mathbb{T}, \mathbb{B})$ over the sample space $\{4, 5, \ldots, 10\} \times \{\mathsf{T}, \mathsf{F}\}$, where $\mathbb{T}$ represents the time I wake up in the morning, and $\mathbb{B}$ represents whether I have breakfast or not. The following table summarizes the joint probability distribution.

t	b	$\mathbb{P}[\mathbb{T} = t, \mathbb{B} = b]$
4	T	0.01
4	F	0.05
5	T	0
5	F	0.04
6	T	0.1
6	F	0.15
7	T	0.30
7	F	0.12
8	T	0.08
8	F	0.05
9	T	0.03
9	F	0.05
10	T	0.01
10	F	0.01

Calculate the following probabilities.

(a) **(5 points)** Calculate the probability that I wake up at 8 a.m. or earlier, but do not have breakfast. That is, calculate $\mathbb{P}[\mathbb{T} \leqslant 8, \mathbb{B} = \mathsf{F}]$,
Solution.

(b) **(5 points)** Calculate the probability that I wake up at 8 a.m. or earlier. That is, calculate $\mathbb{P}\left[\mathbb{T} \leqslant 8\right]$,
Solution.

(c) **(5 points)** Calculate the probability that I skip breakfast conditioned on the fact that I woke up at 8 a.m. or earlier. That is, compute $\mathbb{P}\left[\mathbb{B} = \mathsf{F} \mid \mathbb{T} \leqslant 8\right]$.
Solution.

4. **Random Walk(20 points).** There is a frog sitting at the origin $(0, 0)$ in the first quadrant of a two-dimensional Cartesian plane. The frog first jumps uniformly at random along the X-axis to some point $(\mathbb{X}, 0)$, where $\mathbb{X} \in \{1, 2, 3, 4, 5, 6\}$. Then, it jumps uniformly at random along the Y-axis to some point $(\mathbb{X}, \mathbb{Y})$, where $\mathbb{Y} \in \{1, 2, 3, 4, 5, 6\}$. So $(\mathbb{X}, \mathbb{Y})$ represents the final position of the frog after these two jumps. Note that $\mathbb{X}$ and $\mathbb{Y}$ are two independent random variables that are uniformly distributed over their respective sample spaces.

   (a) **(5 points)** What is the probability that the frog jumps more than 4 units along the Y-axis. That is, compute $\mathbb{P}[\mathbb{Y} > 4]$.
   Solution.

   (b) **(5 points)** What is the probability that the Euclidean distance of the final position of the frog from the origin is at least 7. That is compute $\mathbb{P}\left[\sqrt{\mathbb{X}^2 + \mathbb{Y}^2} \geqslant 7\right]$?
   Solution.

(c) **(10 points)** What is the probability that the frog has jumped at least 5 units along X-axis conditioned on the fact that the distance of the final position of the frog from the origin is at least 7? That is, compute $\mathbb{P}\left[\mathbb{X} \geqslant 5 \mid \sqrt{\mathbb{X}^2 + \mathbb{Y}^2} \geqslant 7\right]$?

5. **Coin Tossing Word Problem(15 points).** We have three (independent) coins represented by random variables $\mathbb{C}_1, \mathbb{C}_2$, and $\mathbb{C}_3$.

   (i) The first coin has $\mathbb{P}[\mathbb{C}_1 = H] = \frac{1}{4}, \mathbb{P}[\mathbb{C}_1 = T] = \frac{3}{4}$,

   (ii) The second coin has $\mathbb{P}[\mathbb{C}_2 = H] = \frac{3}{4}$ and $\mathbb{P}[\mathbb{C}_2 = T] = \frac{1}{4}$, and

   (iii) The third coin has $\mathbb{P}[\mathbb{C}_3 = H] = \frac{1}{4}$ and $\mathbb{P}[\mathbb{C}_3 = T] = \frac{3}{4}$.

   Consider the following experiment.

   (A) Toss the first coin. Let the outcome of the first coin-toss be ω_1.

   (B) If $\omega_1 = H$, then we toss the second coin twice. Otherwise, (i.e., if $\omega_1 = T$) toss the second coin once and then toss the third coin once. Let the two outcomes of this step be represented by ω_2 and ω_3.

   (C) Output $(\omega_1, \omega_2, \omega_3)$.

   Based on this experiment, compute the probabilities below.

   (a) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes $(\omega_1, \omega_2, \omega_3)$ are H (head)?
   Solution.

(b) **(10 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes are H, conditioned on the fact that the first outcome was T?
Solution.

6. **(10 points)** Use the fact that $\exp(-x) \approx 1 - x$ (when x is small) to show

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{t-1}{n}\right) \approx \left(1-\frac{(t-1)t}{2n}\right)$$

when t^2/n is small.

(*Remark*: You shall see the usefulness of this estimation in the topic "Birthday Paradox" that we shall cover in the forthcoming lectures.)